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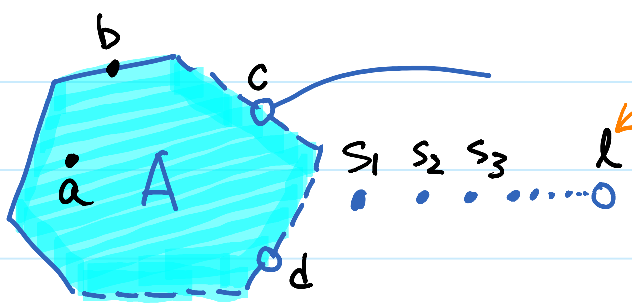
Target of topology in analysis:

limit, convergence, approximation

Need to do it by open sets/neighborhood

Let us consider a set A in $\mathbb{R}^2$



limit of $s_n \in A$ may or may not be in A

Among the points, a, b, c, d, s_{99}, ℓ

one is different from others when

we consider the neighborhoods of each point.

Ah!! s_{99} has small nbhds that does not

meet other points of A

Definition

$x \in A$ is an isolated point of A if

$$\underbrace{\exists \text{ nbhd } N \text{ of } x,}_{\text{can be } U \in \mathcal{T} \text{ and } x \in U} \quad N \cap A \setminus \{x\} = \phi$$

Its negation is expressed as

$$\forall \text{ nbhd } N \text{ of } x, \ N \cap A \setminus \{x\} \neq \phi$$

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Definition

$x \in X$ (not necessarily in A) is a

cluster point
accumulation point
limit point

of A if

$\forall$ nbhd N of x, $N \cap A \setminus \{x\} \neq \emptyset$

$A' = \{$all cluster points of $A\}$ is called the derived set of A

$\bar{A}$ or $Cl(A) = A \cup A'$ is called the closure of A

Intuitively, $\bar{A}$ contains
(i) points of A and
(ii) points stick to A (its skin)

Exercise. $x \in \bar{A} \iff$

$\underbrace{\forall \text{ nbhd } N \text{ of } x}_{\text{can be } \mathcal{U} \in \mathcal{I} \text{ and } x \in \mathcal{U}}$, $N \cap A \neq \emptyset$

Let us think about the negation of above.

That is $x \notin \bar{A} \iff$

$\exists$ nbhd N of x $N \cap A = \emptyset$ $N \subset X \setminus A$

Thus $X \setminus \bar{A} = (X \setminus A)^{\circ}$

Definition

$x \in A$ is called a frontier point of A
boundary point

if $x \in \bar{A} \cap \overline{(X \setminus A)}$.

The frontier of A is denoted $\mathrm{Frt}(A)$.

Obviously, $x \in \mathrm{Frt}(A) \iff$

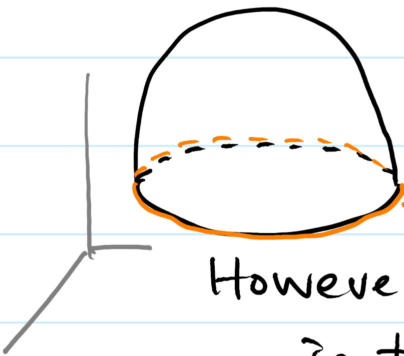
$$\underbrace{\forall \text{ nbhd } N \text{ of } x,}_{U \in \mathcal{T} \text{ and } x \in U} \ N \cap A \neq \emptyset, \ N \cap (X \setminus A) \neq \emptyset$$

Exercise Find $\mathrm{Frt}(A)$ for any $A \subset X$

if (a) $\mathcal{T} = \mathcal{P}(X)$ (b) $\mathcal{T} = \{\emptyset, X\}$

From this exercise, you will see the difference between frontier and "skin".

Remark. Why use frontier not boundary



$A =$ Hemisphere $\subset \mathbb{R}^3$, $\mathcal{T}_{std}$

$\mathrm{Frt}(A) = A$

However boundary $=$ circle

in the sense of surface/manifold

Notation is ∂A

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Definition $A \subset X$ is closed if its complement $X \setminus A$ is open, i.e., $X \setminus A \in \mathcal{J}$

Easy Observation.

ϕ, X are closed because

$\phi = X \setminus X$, $X = X \setminus \phi$ and $X, \phi \in \mathcal{J}$ is known

Thus, ϕ and X are both open and closed.

There may be others depending on $\mathcal{J}$.

Obviously, some are neither open nor closed.

Recall there are 2 statements about open sets versus interior

There are corresponding statements for closed sets

Propositions

1. $F \subset X$ is closed $\Leftrightarrow F = \overline{F}$

$\Leftrightarrow F \supset \overline{F} \Leftrightarrow F \supset F'$

Note: $\overline{F} = F \cup F'$

2. $\overline{F}$ is the smallest closed set containing F

For the proofs, key ingredient $X \setminus \overline{F} = (X \setminus F)^\circ$

F is closed $\Leftrightarrow X \setminus F$ is open

$\Leftrightarrow X \setminus F = (X \setminus F)^\circ = X \setminus \overline{F} \Leftrightarrow F = \overline{F}$

The other statement follows that $\overline{F} = X \setminus (X \setminus F)^\circ$ is closed and $(X \setminus F)^\circ$ is largest

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Think about possibilities of topology for X

Largest $\mathcal{T} = \mathcal{P}(X)$, discrete

Smallest $\mathcal{T} = \{\phi, X\}$, indiscrete

① Suppose we know $\phi \neq A \subsetneq X$ and $A \in \mathcal{T}$

What is the smallest $\mathcal{T}$?

$\{\phi, A, X\}$ is it

② If we have $A, B \in \mathcal{T}$, $\phi \neq \begin{matrix} A \\ \| \\ B \end{matrix} \subsetneq X$ then

smallest $\mathcal{T} = \{\phi, A \cap B, A, B, A \cup B, X\}$

Given $\mathcal{S} \subset \mathcal{P}(X)$, how to get a minimal topology $\mathcal{T}$ that $\mathcal{T} \supset \mathcal{S}$

Definition

The smallest topology containing $\mathcal{S}$ is the topology generated by $\mathcal{S}$.

One may get it simply by brute force, i.e, try all combinations of arbitrary union and finite intersection.

Theorem. Given any $\mathcal{S} \subset \mathcal{P}(X)$, let

$$\mathcal{B} = \{\bigcap \mathcal{F} : \text{finite } \mathcal{F} \subset \mathcal{S}\} = \{S_1 \cap \cdots \cap S_n : S_k \in \mathcal{S}\}$$

$$\mathcal{T} = \{\bigcup \mathcal{A} : \text{any } \mathcal{A} \subset \mathcal{B}\} = \{\bigcup_{\alpha \in I} B_\alpha : B_\alpha \in \mathcal{B}\}$$

Then $\mathcal{T}$ is the smallest topology containing $\mathcal{S}$

or $\mathcal{T}$ is generated by $\mathcal{S}$

or $\mathcal{S}$ is a subbase (subbasis) of $\mathcal{T}$

Definition

$\mathcal{B} \subset$ topology $\mathcal{T}$ is a base if

$\mathcal{T}$ can be formed by taking unions from $\mathcal{B}$

i.e. $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$

Example. For $X = \mathbb{R}$, let

$$\mathcal{S} = \{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$$

After taking finite intersections

$$\mathcal{B} = \mathcal{S} \cup \{(a,b) : a < b \in \mathbb{R}\} \cup \{X\} \cup \{\emptyset\}$$

Then, obviously, after arbitrary unions,

we get the standard topology of $\mathbb{R}$.

Lower Limit Topology of $\mathbb{R}$

$\mathcal{T}_{ll}$ is the topology generated by

$$\{[a,b) : a<b\in\mathbb{R}\}$$

Note that $\mathcal{T}_{std} \subset \mathcal{T}_{ll}$

because $\bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b) = (a,b)$

Exercise. Find a sequence $x_n \in \mathbb{R}$ such that

① $x_n \longrightarrow x$, i.e., $\forall\ \varepsilon>0\ \exists$ integer N that $\forall\ n\geq N\ \ x_n \in (x-\varepsilon, x+\varepsilon)$, but

② $\exists\ \delta>0$ that $\forall$ integer $N\ \exists\ n\geq N$ $x_n \notin [x, x+\delta)$

This example shows that

$x_n \longrightarrow x$ in $\mathcal{T}_{std}$ but $x_n \not\longrightarrow x$ in $\mathcal{T}_{ll}$